Speranza matematica (o media)

Si lancia un dado equilibrato

$X =$ n° che esce sul dado

$$\frac{1+2+3+4+5+6}{6} = \frac{7}{2} = 3.5$$

2° caso

1 esce con prob. doppia delle altre facce

$1 \to \frac{2}{7}$ $\qquad \frac{2}{7} + \frac{1}{7} \times 5 = \frac{7}{7} = 1$

$2 \to \frac{1}{7}$

$\vdots \qquad \vdots \qquad 1$

$6 \to \frac{1}{7}$

$$\to 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6} = 3.5$$

$$1 \times \frac{2}{7} + 2 \times \frac{1}{7} + 3 \times \frac{1}{7} + \dots \quad 6 \times \frac{1}{7} = \dots$$

$$X = \begin{cases} 1 \longrightarrow 1/6 = P(X=1) \\ 2 \longrightarrow 1/6 = P(X=2) \\ 3 \longrightarrow 1/6 \\ 4 \longrightarrow 1/6 \\ 5 \longrightarrow 1/6 \\ 6 \longrightarrow 1/6 \end{cases} \qquad X = \begin{cases} 1 \to 2/7 \\ 2 \to 1/7 \\ 3 \to 1/7 \\ 4 \to 1/7 \\ 5 \to 1/7 \\ 6 \to 1/7 \end{cases}$$

$$\sum_x x \, P(X=x) = \sum_x x \, p(x)$$

$$p(x) = P(X=x)$$

Sia X una v. a. discreta con dens. p

Definizione Si dice che X ha speranza matematica finita

se $\sum_x |x| \, p(x) < \infty$

e in tal caso, si chiama speranza (matematica) di X (media, valor medio) (in inglese expectation, mean) il numero

$$E[X] = \sum_x x \, p(x)$$

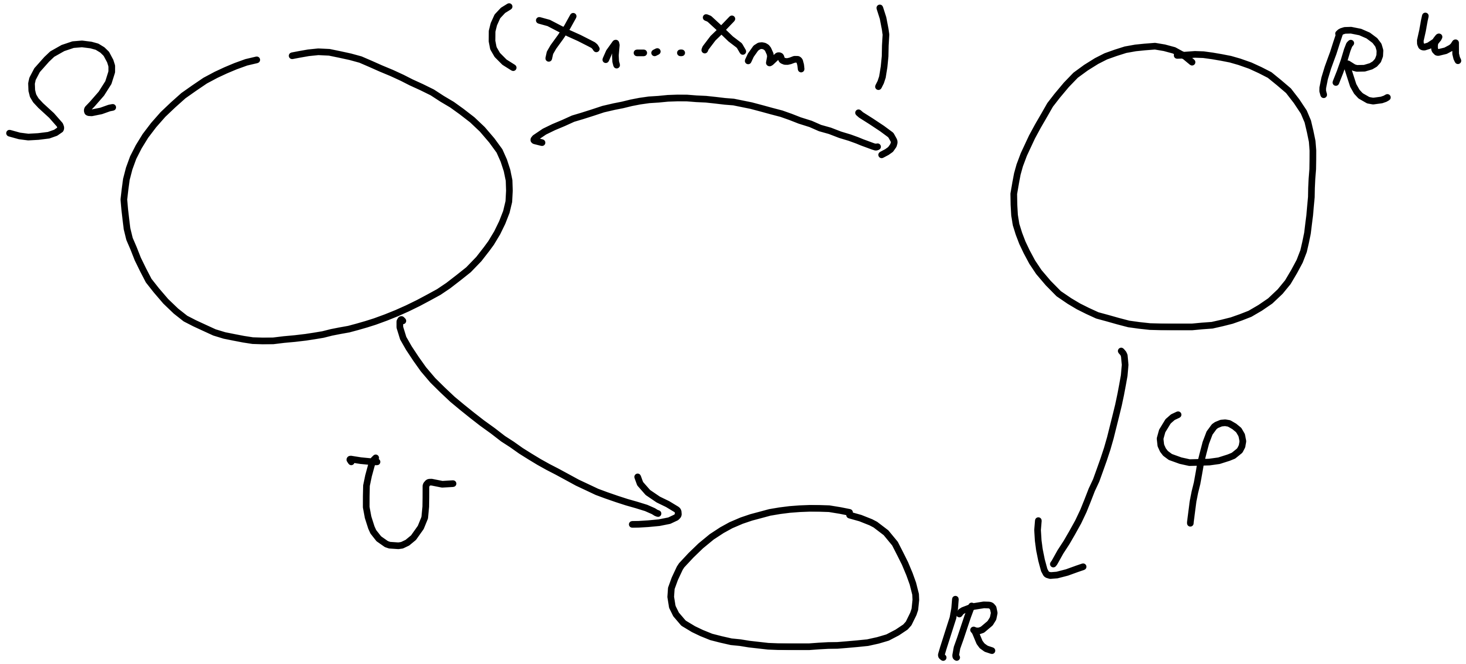
$$\sum_x \left| x \; p(x) \right| = \sum |x| \; p(x) < \infty$$

$$\boxed{\sum_x x \, \underbrace{p(x)}_{= P(X=x)} = E[X]}$$

X

<u>Teorema.</u> Sia $(X_1, \dots, X_m)$ un vettore di m v. a. discrete con dens. cong $p(x_1, \dots, x_m)$; e sia $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}$ una funzione

Cons. la v. a. $\underline{U} = \underline{\varphi}(\underline{X_1, \dots, X_m})$



Allora U ha speranza finita

se e solo se

$$\sum_{x_1, \dots x_m} |\varphi(x_1, \dots, x_m)| \, p(x_1, \dots, x_m) < \infty$$

e in tal caso

$$E[U] = \sum_{x_1 \dots x_m} \underline{\varphi}(x_1, \dots, x_m) \, \underline{p(x_1 \dots x_m)}$$

$$\sum_u u \, \underline{P} (U = u)$$

<u>Proposizione.</u> Siano X e Y 2 v. a. discrete, entrambe con speranza finita. Allora

1) La v.a. cX ha speranza finita e $E[cX] = c\,E[X]$ $\quad (c \in \mathbb{R})$

2) La v.a. $\underline{Z} = X+Y$ ha speranza finita e

$$\underline{E[X+Y]} = \underline{E}[X] + \underline{E}[Y]$$

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$$

Dim. di 2)

$$Z = X+Y$$
$$Z = \varphi(X,Y)$$
$$\varphi(x,y) = x+y$$

Devo far vedere che

$$\sum_{x,y}' |\varphi(x,y)|\, p(x,y) < \infty, \text{ dove}$$

$p(x,y)$ è la dens. cong. di (X,Y)

$$p(x,y) = P(X=x,\ Y=y)$$

$$0 \le \sum_{x,y}' |x+y|\, p(x,y) \le \sum_{x,y}' (|x|+|y|)\, p(x,y)$$

$$= \sum_{x,y}' |x|\, p(x,y) + \sum_{x,y}' |y|\, p(x,y) =$$

$$\sum_{x}' |x| \sum_{y}' p(x,y) + \sum_{y}' |y| \sum_{x}' p(x,y)$$

$P(X=x)$

$$= \underbrace{\sum_{x}' |x|\, p_X(x)}_{<\infty} + \underbrace{\sum_{y}' |y|\, p_Y(y)}_{<\infty}$$

$$E[Z] = E[\varphi(X,Y)] = \sum_{x,y} \varphi(x,y)\, p(x,y) =$$

$$= \sum_{x,y} (x+y)\, p(x,y) = \sum_{x,y} x\, p(x,y) +$$

$$+ \sum_{x,y} y\, p(x,y) = \sum_{x} x \sum_{y} p(x,y) +$$

$$+ \sum_{y} y \sum_{x} p(x,y) =$$

$$= \underbrace{\sum_{x} x\, p_X(x)} + \underbrace{\sum_{y} y\, p_Y(y)}$$

$$= E[X] + E[Y]$$

Proposizione. Siano X e Y 2 v.a. discrete, entrambe con speranza finita. Allora

1) Se $P(X \leq Y) = 1$, allora

$$E[X] \leq E[Y]$$

2) $$|E[X]| \leq E[|X|]$$

Dim. di 2) $-|X| \leq X \leq |X|$

$$E[-|X|] \leq E[X] \leq E[|X|]$$

$$-\underbrace{E[|X|]}_{a} \leq \underbrace{E[X]}_{b} \leq \underbrace{E[|X|]}_{a}$$

$$-a \leq b \leq a \iff |b| \leq a$$

1) $Z = Y - X$

$E[X] \le E[Y]$

$E[Y] - E[X] \ge 0$

$E[Y - X] \ge 0$

$E[Z] \ge 0$

Hp: $P(Z \ge 0) = 1$ $\Leftarrow$

Th $E[Z] \ge 0$ $\Downarrow$ $\Updownarrow$

$E[Z] = \sum_z \underbrace{z}_{\ge 0} \underbrace{P(Z = z)}_{\ge 0}$

≥ 0

$$E[|X|] = \sum_x \varphi(x) P(X=x) = \qquad |X| = \varphi(X) \quad \varphi(t) = |t|$$

$$= \sum_x \varphi(x)\, p(x) = \sum_x |x|\, p(x)$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[XY] \stackrel{?}{=} E[X] \cdot E[Y] \quad \text{no}$$

Proposizione. Siano X e Y 2 v.a. discrete con dens. $p(x,y)$ aventi ~~s~~ entrambe speranza finita. Se X e Y sono indipendenti, allora la v.a. $Z = XY$ ha speranza finita e

$$\longrightarrow E[XY] = E[X] \cdot E[Y]$$

Dimostrazione: $Z = \varphi(X,Y)$ $\quad \varphi(x,y) = xy$

$$\sum_{x,y} |xy|\, p(x,y) = \sum_{x,y} |x||y|\, \underbrace{p(x,y)}_{p_X(x)\cdot p_Y(y)} =$$

$$= \sum_x |x| \sum_y |y|\, p(x,y) =$$

$$= \underbrace{\left(\sum_x |x|\, p_X(x)\right)}_{<\infty} \underbrace{\left(\sum_y |y|\, p_Y(y)\right)}_{<\infty}$$

$$E[Z] = E[\varphi(X,Y)] =$$

$$= \sum_{x,y} \varphi(x,y)\, p_X(x)\, p_Y(y) =$$

$$= \sum_{x,y} (x \cdot y)\big(p_X(x)\, p_Y(y)\big) =$$

$$= \underbrace{\left(\sum_x x\, p_X(x)\right)}_{} \cdot \left(\sum_y y\, p_Y(y)\right)$$

$$= E[X] \cdot E[Y]$$

Esempi:

1) $P(X = c) = 1$ $\qquad E[X] = c \cdot P(X=c) = c \cdot 1 = c$

2) $X = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$ $\qquad X \sim B(1, p)$

$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$

3) $X \sim B(n, p)$

$$E[X] = \sum_x x f_X(x) = \qquad f_X(x) \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0 \ldots n \\ 0 & - \end{cases}$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} =$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} =$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} =$$

$$y = x - 1 \qquad x = y + 1$$

$$= \sum_{y=0}^{n-1} \frac{n!}{y!(n-y-1)!} p^{y+1} (1-p)^{n-y-1} =$$

$$= (np) \underbrace{\sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!} p^y (1-p)^{n-1-y}}_{B(n-1,\, p)}$$

$$= (np) \cdot 1 = np$$

$$X \sim B(n, p)$$

$$\rightarrow X = \sum_{i=1}^{n} X_i \qquad X_i \sim B(1,p)$$

$$X_i = \begin{cases} 1 & \text{se succ. all'i-esima prova} \\ 0 & \text{—} \end{cases}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] =$$

$$= \sum_{i=1}^{n} \underbrace{E[X_i]} = \sum_{i=1}^{n} p = np$$

$X \sim$ geom. di par. p.

$$f(x) = P(X = x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \dots \\ 0 & - \end{cases}$$

$$\sum_x x\, f(x) = \sum_x x\, f(x) =$$

$$= \sum_{x=1}^{\infty} x\, p\, (1-p)^{x-1} = \frac{1}{p}$$

$$* = -p \sum_{x=1}^{\infty} \frac{d}{dp} (1-p)^x =$$

$$= -p \frac{d}{dp} \left(\sum_{x=1}^{\infty} (1-p)^x \right)$$

$$= -p \frac{d}{dp} \left(\frac{1}{1-(1-p)} - 1 \right) = \dots \frac{1}{p}$$

4) $X \sim \Pi_\lambda$

$$p(x) = \begin{cases} \frac{\lambda^x}{x!} e^{-\lambda} & x = 0, 1, 2 \dots \\ 0 & \text{---} \end{cases}$$

$$\sum_{x} x\, p(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} =$$

$$= \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} e^{-\lambda} =$$

$x - 1 = y$

$x = 1 + y$

$$= \sum_{y=0}^{\infty} \frac{\lambda^{1+y}}{y!} e^{-\lambda} = \lambda \underbrace{\left(\sum_{y=0}^{\infty} \frac{\lambda^y}{y!} e^{-\lambda} \right)}_{=1}$$

$\lambda = 3$

$$= \lambda \cdot 1 = \lambda$$

$$E[X+Y] = E[X]+E[Y]$$

$$E[(X+Y)+Z] = E[X+Y]+E[Z]$$

$$= E[X]+E[Y]+E[Z]$$